Time response of Second-order systems

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General formula of a second-order system transfer function can be written as

$$F(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (1)$$

where $\omega_n = 2 \cdot \pi \cdot f_n$ (where f_n is a system cut-off frequency) and ζ is called a dumping factor. Time response of second-order systems can be divided into three important cases depends on the dumping factor ζ value.

1 $\zeta > 1$ - overdamped system

For $\zeta > 1$ two real poles $p_1 = \omega_n(\zeta - \sqrt{\zeta^2 - 1})$ and $p_2 = \omega_n(\zeta + \sqrt{\zeta^2 - 1})$ can be found for the denominator of transfer function (1). Thus step response of overdamped system can be written as

$$P(s) = \frac{1}{s} \frac{\omega_n^2}{(s + p_1)(s + p_2)}. \quad (2)$$

Step response (2) can be resolved as

$$P(s) = \frac{1}{s} - \frac{p_2}{p_2 - p_1} \frac{1}{s + p_1} + \frac{p_1}{p_2 - p_1} \frac{1}{s + p_2} \quad (3)$$

which gives time domain formula for overdamped system

$$p(t) = 1 - \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1\right) e^{-\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} + \quad (4)$$
$$+ \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - 1}} - 1\right) e^{-\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)t}$$

Using the Laplace transform of Dirac delta $L\{\delta(t)\} = 1$ instead of step Laplace transform s^{-1} in equation (2) the pulse response can be written as

$$D(s) = \frac{\omega_n^2}{(s + p_1)(s + p_2)} \quad (5)$$

which gives the pulse response in time domain of overdamped system as

$$d(t) = \frac{1}{2\omega_n\sqrt{\zeta^2 - 1}} \left[e^{-\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} - e^{-\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)t} \right]. \quad (6)$$

2 $\zeta = 1$ - critically damped system

For $\zeta = 1$ second order pole $p_0 = \omega_n^2$ can be found for transfer function (1) and step response of critically damped system is

$$P(s) = \frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2} \tag{7}$$

which can be rewritten as

$$P(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \tag{8}$$

which gives the step response in time domain

$$p(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}. \tag{9}$$

The pulse response given by formula

$$D(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \tag{10}$$

can be easily resolved in time domain as

$$d(t) = te^{-\omega_n t}. \tag{11}$$

3 $\zeta < 1$ - underdamped system

For the underdamped system with $\zeta < 1$ two complex conjugate poles for the denominator of transfer function (1) can be found. In this case the step response

$$P(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12}$$

can be rewritten as

$$P(s) = \frac{1}{s} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{13}$$

and presented in time domain by formula

$$p(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \psi\right) \tag{14}$$

where ω_d, called damped frequency, is given by

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{15}$$

and the phase shift ψ can be calculated from dumping factor ζ using equation

$$\psi = \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}. \tag{16}$$

The pulse response of underdamped system given directly by formula (1) can be resolved in time domain as

$$d(t) = \frac{\omega_n^2}{\omega_d} e^{-\zeta \omega_n t} \sin \omega_d t. \tag{17}$$

4 Example of step and pulse response of the second order system

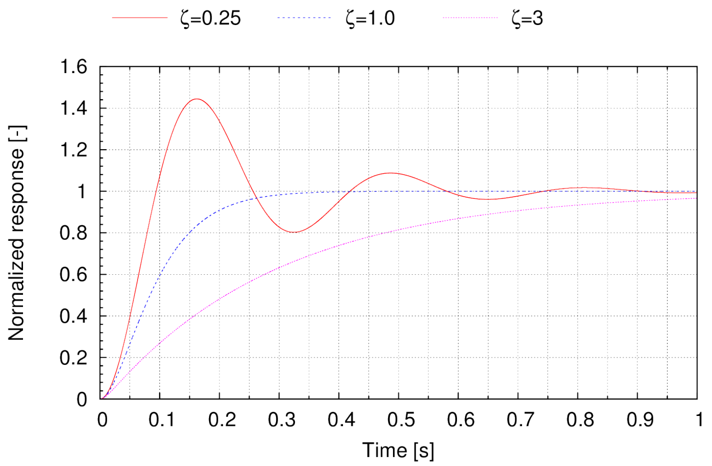

Figure 1: Step response of the second orded system with $\omega_n = 20$ for different dumping factors ζ

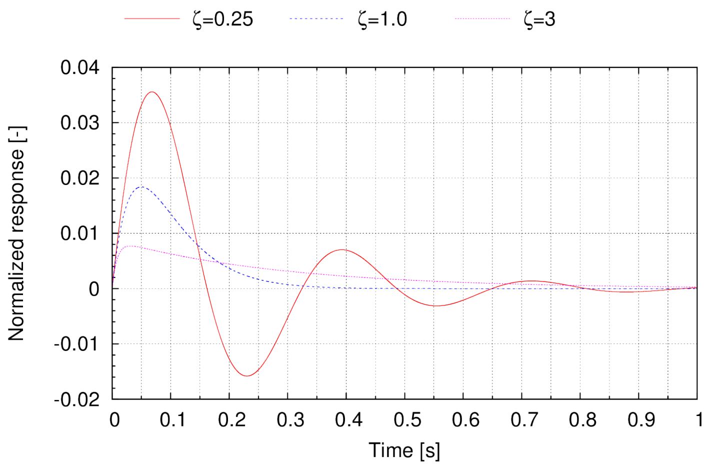

Figure 2: Pulse response of the second orded system with $\omega_n = 20$ for different dumping factors ζ